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Virasoro character identities from the Andrews–Bailey construction

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<u>ABSTRACT</u>

We prove q-series identities between bosonic and fermionic representations of certain Virasoro characters. These identities include some of the conjectures made by the Stony Brook group as special cases. Our method is a direct application of Andrews' extensions of Bailey's lemma to recently obtained polynomial identities.

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1 Introduction

1.1 Aim

In an impressive series of papers that include [1], the Stony Brook group conjectured a large number of q-series identities. For reviews and complete references, see [2]. Let us restrict our attention to those conjectures related to the following Virasoro characters [1]:

(i) $\chi_{r,s}^{(p,p+1)}(q)$ of the unitary minimal model $\mathcal{M}(p,p+1)$, for any r and s;

(ii) $\chi_{(p-1)/2,(p+1)/2}^{(p,p+2)}(q)$ of the non-unitary minimal model $\mathcal{M}(p,p+2)$, where p is odd;

(iii) $\chi_{1,k}^{(p,kp+1)}(q)$ of the non-unitary minimal model $\mathcal{M}(p,kp+1)$ where $p \geq 3$ and $k \geq 1$.

Here, $\mathcal{M}(p,p')$ is a Virasoro minimal model specified by two coprime integers (p,p'), and $\chi_{r,s}^{(p,p')}(q)$ is a conformal character of irreducible highest weight representation of $\mathcal{M}(p,p')$, specified by two integers (r,s), where $1 \leq r \leq p-1$, $1 \leq s \leq p'-1$ [3].

These identities are of great interest for a number of physical and mathematical reasons which are beyond the scope of this work. The first step towards proving them was taken by Melzer [4], who conjectured a polynomial identity which implies the q-series identities for (i) in the above list, and proved it for $p = 3, 4$. In [5] Berkovich proved these polynomial identities for arbitrary p and for $s = 1$, and thus proved the q-series identities involving (i) for $s = 1$. In [6] we presented a polynomial identity which implies Gordon's generalization of the Rogers–Ramanujan identities [7], and the q-series identities for $\chi_{1,i}^{(2,2k+1)}(q)$ [8, 9].

In this paper, we prove q-series identities for

(II) $\begin{cases} \chi_{\iota,k\kappa+1}^{(2\kappa+1,k(2\kappa+1)+2)}(q) \text{ and } \chi_{\iota,(k+1)\kappa+1}^{(2\kappa+1,k(2\kappa+1)+2)}(q) \\ \chi_{\iota,k\kappa+k-1}^{(2\kappa+1,k(2\kappa+1)+2\kappa-1)}(q) \text{ and } \chi_{\iota,(k+1)\kappa+k-1}^{(2\kappa+1,k(2\kappa+1)+2\kappa-1)}(q) \end{cases}$ for $1 \leq \iota \leq \kappa$.

(III) $\begin{cases} \chi_{1,r(k+1)}^{(p,kp+p-1)}(q) \text{ and } \chi_{1,r(k+1)+k}^{(p,kp+p-1)}(q) \\ \chi_{1,kr}^{(p,kp+1)}(q) \text{ and } \chi_{1,k(r+1)+1}^{(p,kp+1)}(q) \end{cases}$ for $p \geq 4$, $1 \leq r \leq p-2$ and $k \geq 1$,

which include (ii) and (iii) as special cases, respectively. In order to obtain the above fermionic representations for (II) and (III), we apply Andrews' extensions of Bailey's lemma [10, 11, 12, 13] to the polynomial identities presented in [6], and [4, 5], respectively.

1.2 Plan

This paper is organized as follows. In the rest of this section, we formulate the problem and summarize our results. In section 2 we review a number of definitions and propositions concerning Andrews' extensions of Bailey's lemma [10, 11, 12, 13]. We wish to refer to these extensions as the *Andrews–Bailey construction*. In section 3 and section 4 we obtain the desired q-series identities related to (II) and (III), respectively. In section 5 we give some remarks. Appendix A contains a summary of some of the q-series identities that can be obtained by a direct application of the Andrews–Bailey construction to Slater's list of Bailey pairs [20].

1.3 Formulation of the problem

Rocha-Caridi [14] obtained the following expression for the Virasoro characters as an infinite series in q:

$$\chi_{r,s}^{(p,p')}(q) = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \left(q^{pp'n^2+(rp'-sp)n} - q^{(pn+r)(p'n+s)} \right), \tag{1.1}$$

where p and p' are coprime positive integers, $1 \le r \le p-1$, $1 \le s \le p'$, and

$$(a)_\infty \equiv (a;q)_\infty = \prod_{m=0}^{\infty} (1 - aq^m).$$

Starting from Bethe ansatz computations, the Stony Brook group [1] found different q-series expressions for large classes of these characters. For physical reasons their new expressions are referred to as the *fermionic* sum representations, whereas (1.1) is referred to as the *bosonic* sum representation.

For non-unitary minimal model $\mathcal{M}(2, 2k+1)$, (1.1) reduces to

$$
\begin{aligned}
\chi_{1,i}^{(2,2k+1)}(q) &= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \left(q^{(4k+2)n^2 + (2k-2i+1)n} - q^{(2n+1)((2k+1)n+i)} \right) \\
&= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{n((2k+1)n+2k-2i+1)/2} \\
&= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm i (\mathrm{mod} 2k+1)}}^{\infty} (1 - q^n)^{-1},
\end{aligned}
\tag{1.2}
$$

where the last equality is obtained using Jacobi's triple product formula.

The fermionic expression corresponding to (1.2), which was obtained in [8, 9] from different approaches, is as follows:

$$
\chi_{1,i}^{(2,2k+1)}(q) = \chi_{1,2k+1-i}^{(2,2k+1)}(q) = \sum_{n_1 \ge \cdots \ge n_{k-1} \ge 0} \frac{q^{n_1^2 + \cdots + n_{k-1}^2 + n_i + \cdots + n_{k-1}}}{(q)_{n_1 - n_2} \cdots (q)_{n_{k-2} - n_{k-1}} (q)_{n_{k-1}}},
\tag{1.3}
$$

where $1 \le i \le k$, and

$$(a)_n \equiv (a;q)_n = \frac{(a)_\infty}{(aq^n)_\infty}.$$

Equating these two expressions, we reproduce Gordon's generalization of the Rogers–Ramanujan identity [7] for $1 \le i \le k$

$$
\prod_{\substack{n=1 \\ n \not\equiv 0, \pm i (\mathrm{mod} 2k+1)}}^{\infty} (1 - q^n)^{-1} = \sum_{n_1 \ge \cdots \ge n_{k-1} \ge 0} \frac{q^{n_1^2 + \cdots + n_{k-1}^2 + n_i + \cdots + n_{k-1}}}{(q)_{n_1 - n_2} \cdots (q)_{n_{k-2} - n_{k-1}} (q)_{n_{k-1}}}.
\tag{1.4}
$$

This is the simplest q-series identity between bosonic and fermionic representations of Virasoro characters.

In [1] the Stony Brook group proved identities of the above type up to a finite power in q, using explicit computations, and conjectured their validity to all powers in q. Proving some of these conjectures is the problem we address in this work [1].

1.4 Summary of results

In section 3 we shall prove the following q-series identities. The notation will be explained in section 3.1.

---

[1] There exists an extensive literature devoted to this topic but not discussed in this paper [15, 16, 17, 18].

Theorem 1.1 *The following identities hold:*

$$
\begin{aligned}
&\frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \Big(q^{(2\kappa+1)((2\kappa+1)k+2)n^2 + (\iota((2\kappa+1)k+2)-(k\kappa+1)(2\kappa+1))n} \\
&- \quad q^{((2\kappa+1)n+\iota)((2\kappa+1)k+2)n)+(k\kappa+1))} \Big) \\
&= \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + (\kappa-\iota)(n_1+\cdots+n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+\kappa-\iota}} \\
&\times \sum_{\substack{\nu_1 \geq \cdots \geq \nu_{\kappa-1} \geq \nu_\kappa = 0 \\ \nu_1 + \cdots + \nu_{\kappa-1} \leq n_k}} q^{\nu_1^2 + \cdots + \nu_{\kappa-1}^2 + \nu_\iota + \cdots + \nu_{\kappa-1}} \\
&\times \prod_{\mu=1}^{\kappa-1} \left[\begin{matrix} 2n_k + \kappa - \iota - 2(\nu_1 + \cdots + \nu_{\mu-1}) - \nu_\mu - \nu_{\mu+1} - \alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu - \nu_{\mu+1} \end{matrix} \right]_q .
\end{aligned} \tag{1.5}
$$

$$
\begin{aligned}
&\frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \Big(q^{(2\kappa+1)((2\kappa+1)k+2)n^2 + (\iota((2\kappa+1)k+2)-(k(\kappa+1)+1)(2\kappa+1))n} \\
&- \quad q^{((2\kappa+1)n+\iota)((2\kappa+1)k+2)n)+(k(\kappa+1)+1))} \Big) \\
&= \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 \cdots + n_k^2 + (\kappa-\iota+1)(n_1+\cdots+n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+\kappa-\iota+1}} \\
&\times \sum_{\substack{\nu_1 \geq \cdots \geq \nu_{\kappa-1} \geq \nu_\kappa = 0 \\ \nu_1 + \cdots + \nu_{\kappa-1} \leq n_k}} q^{\nu_1^2 + \cdots + \nu_{\kappa-1}^2 + \nu_\iota + \cdots + \nu_{\kappa-1}} \\
&\times \prod_{\mu=1}^{\kappa-1} \left[\begin{matrix} 2n_k + \kappa - \iota + 1 - 2(\nu_1 + \cdots + \nu_{\mu-1}) - \nu_\mu - \nu_{\mu+1} - \alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu - \nu_{\mu+1} \end{matrix} \right]_q .
\end{aligned} \tag{1.6}
$$

As a corollary we have

Corollary 1.2

$$
\begin{aligned}
\chi_{\iota,k\kappa+1}^{(2\kappa+1,(2\kappa+1)k+2)}(q) &= \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + (\kappa-\iota)(n_1+\cdots+n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+\kappa-\iota}} \\
&\times \sum_{\substack{\nu_1 \geq \cdots \geq \nu_{\kappa-1} \geq \nu_\kappa = 0 \\ \nu_1 + \cdots + \nu_{\kappa-1} \leq n_k}} q^{\nu_1^2 + \cdots + \nu_{\kappa-1}^2 + \nu_\iota + \cdots + \nu_{\kappa-1}} \\
&\times \prod_{\mu=1}^{\kappa-1} \left[\begin{matrix} 2n_k + \kappa - \iota - 2(\nu_1 + \cdots + \nu_{\mu-1}) - \nu_\mu - \nu_{\mu+1} - \alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu - \nu_{\mu+1} \end{matrix} \right]_q .
\end{aligned} \tag{1.7}
$$

$$
\begin{aligned}
\chi_{\iota,k(\kappa+1)+1}^{(2\kappa+1,(2\kappa+1)k+2)}(q) &= \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 \cdots + n_k^2 + (\kappa-\iota+1)(n_1+\cdots+n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+\kappa-\iota+1}} \\
&\times \sum_{\substack{\nu_1 \geq \cdots \geq \nu_{\kappa-1} \geq \nu_\kappa = 0 \\ \nu_1 + \cdots + \nu_{\kappa-1} \leq n_k}} q^{\nu_1^2 + \cdots + \nu_{\kappa-1}^2 + \nu_\iota + \cdots + \nu_{\kappa-1}} \\
&\times \prod_{\mu=1}^{\kappa-1} \left[\begin{matrix} 2n_k + \kappa - \iota + 1 - 2(\nu_1 + \cdots + \nu_{\mu-1}) - \nu_\mu - \nu_{\mu+1} - \alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu - \nu_{\mu+1} \end{matrix} \right]_q .
\end{aligned} \tag{1.8}
$$

Setting $k = 1, \iota = \kappa = (p-1)/2$ and $2n_1 = m_1, 2\nu_1 = m_1 - m_2, \cdots, 2\nu_{\kappa-1} = m_{\kappa-1} - m_\kappa$ in (1.7), we reproduce the corresponding expressions in [1].

Theorem 1.3 *The following identities hold:*

$$
\begin{aligned}
\frac{1}{(q)_\infty} &\sum_{n=-\infty}^{\infty} \Big(q^{(2\kappa+1)((2\kappa+1)k+2\kappa-1)n^2+(\iota((2\kappa+1)k+2\kappa-1)-(k\kappa+k-1)(2\kappa+1))n} \\
&\quad - q^{((2\kappa+1)n+\iota)((2\kappa+1)k+2\kappa-1)n+)(k\kappa+k-1)} \Big) \\
&= \sum_{n_1\geq\cdots\geq n_k\geq 0} \frac{q^{n_1^2+\cdots+n_{k-1}^2+2n_k^2+(\kappa-\iota)(n_1+\cdots+n_{k-1}+2n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+\kappa-\iota}} \\
&\quad\times \sum_{\substack{\nu_1\geq\cdots\geq\nu_{\kappa-1}\geq\nu_\kappa=0 \\ \nu_1+\cdots+\nu_{\kappa-1}\leq n_k}} q^{\nu_1^2+\cdots+\nu_{\kappa-1}^2-\nu_1(2n_k+\kappa-\iota)} \times \\
&\quad\times \prod_{\mu=1}^{\kappa-1} \begin{bmatrix} 2n_k+\kappa-\iota-2(\nu_1+\cdots+\nu_{\mu-1})-\nu_\mu-\nu_{\mu+1}-\alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu-\nu_{\mu+1} \end{bmatrix}_q.
\end{aligned}
\tag{1.9}
$$

$$
\begin{aligned}
\frac{1}{(q)_\infty} &\sum_{n=-\infty}^{\infty} \Big(q^{(2\kappa+1)((2\kappa+1)k+2\kappa-1)n^2+(\iota((2\kappa+1)k+2\kappa-1)-(k\kappa+k+\kappa)(2\kappa+1))n} \\
&\quad - q^{((2\kappa+1)n+\iota)((2\kappa+1)k+2\kappa-1)n+k\kappa+k+\kappa)} \Big) \\
&= \sum_{n_1\geq\cdots\geq n_k\geq 0} \frac{q^{n_1^2\cdots+n_{k-1}^2+2n_k^2+(\kappa-\iota+1)(n_1+\cdots+n_{k-1}+2n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+\kappa-\iota+1}} \\
&\quad\times \sum_{\substack{\nu_1\geq\cdots\geq\nu_{\kappa-1}\geq\nu_\kappa=0 \\ \nu_1+\cdots+\nu_{\kappa-1}\leq n_k}} q^{\nu_1^2+\cdots+\nu_{\kappa-1}^2-\nu_1(\kappa-\iota+1)} \\
&\quad\times \prod_{\mu=1}^{\kappa-1} \begin{bmatrix} 2n_k+\kappa-\iota+1-2(\nu_1+\cdots+\nu_{\mu-1})-\nu_\mu-\nu_{\mu+1}-\alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu-\nu_{\mu+1} \end{bmatrix}_q.
\end{aligned}
\tag{1.10}
$$

As a corollary we have

Corollary 1.4

$$
\begin{aligned}
\chi_{\iota,\kappa k+\kappa-1}^{(2\kappa+1,(2\kappa+1)(k+1)-2)}(q) &= \sum_{n_1\geq\cdots\geq n_k\geq 0} \frac{q^{n_1^2+\cdots+n_{k-1}^2+2n_k^2+(\kappa-\iota)(n_1+\cdots+n_{k-1}+2n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+\kappa-\iota}} \\
&\quad\times \sum_{\substack{\nu_1\geq\cdots\geq\nu_{\kappa-1}\geq\nu_\kappa=0 \\ \nu_1+\cdots+\nu_{\kappa-1}\leq n_k}} q^{\nu_1^2+\cdots+\nu_{\kappa-1}^2-\nu_1(2n_k+\kappa-\iota)} \times \\
&\quad\times \prod_{\mu=1}^{\kappa-1} \begin{bmatrix} 2n_k+\kappa-\iota-2(\nu_1+\cdots+\nu_{\mu-1})-\nu_\mu-\nu_{\mu+1}-\alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu-\nu_{\mu+1} \end{bmatrix}_q.
\end{aligned}
\tag{1.11}
$$

$$
\begin{aligned}
\chi_{\iota,k(\kappa+1)+\kappa}^{(2\kappa+1,(2\kappa+1)k+2\kappa-1)}(q) &= \sum_{n_1\geq\cdots\geq n_k\geq 0} \frac{q^{n_1^2\cdots+n_{k-1}^2+2n_k^2+(\kappa-\iota+1)(n_1+\cdots+n_{k-1}+2n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+\kappa-\iota+1}} \\
&\quad\times \sum_{\substack{\nu_1\geq\cdots\geq\nu_{\kappa-1}\geq\nu_\kappa=0 \\ \nu_1+\cdots+\nu_{\kappa-1}\leq n_k}} q^{\nu_1^2+\cdots+\nu_{\kappa-1}^2-\nu_1(\kappa-\iota+1)} \\
&\quad\times \prod_{\mu=1}^{\kappa-1} \begin{bmatrix} 2n_k+\kappa-\iota+1-2(\nu_1+\cdots+\nu_{\mu-1})-\nu_\mu-\nu_{\mu+1}-\alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu-\nu_{\mu+1} \end{bmatrix}_q.
\end{aligned}
\tag{1.12}
$$

In section 4 we prove the following q-series identities. The notation will be explained in section 4.1.

Theorem 1.5

$$\frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \left(q^{p(kp+p-1)n^2+(kp+p-1-r(k+1)p)n} - q^{(pn+1)((kp+p-1)n+r(k+1))} \right)$$

$$= \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2+\cdots+n_k^2+(r-1)(n_1+\cdots+n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+r-1}} \tag{1.13}$$

$$\times \sum_{\mathbf{m}\in 2\mathbb{Z}^n+\mathbf{Q}_{p-1-r,p-1}} q^{(\mathbf{m},C_{p-3}\mathbf{m})/4} \prod_{a=1}^{p-3} \left[\begin{array}{c} (\mathbf{e}_a, I_{p-3}\mathbf{m} + \mathbf{e}_{p-1-r} + (2n_k+r-1)\mathbf{e}_1)/2 \\ (\mathbf{e}_a, \mathbf{m}) \end{array} \right]_q .$$

$$\frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \left(q^{p(kp+p-1)n^2+(kp+p-1-(rk+k+r)p)n} - q^{(pn+1)((kp+p-1)n+rk+k+r)} \right)$$

$$= \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2+\cdots+n_k^2+r(n_1+\cdots+n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+r}} \tag{1.14}$$

$$\times \sum_{\mathbf{m}\in 2\mathbb{Z}^n+\mathbf{Q}_{r,1}} q^{(\mathbf{m},C_{p-3}\mathbf{m})/4} \prod_{a=1}^{p-3} \left[\begin{array}{c} (\mathbf{e}_a, I_{p-3}\mathbf{m} + \mathbf{e}_r + (2n_k+r)\mathbf{e}_1)/2 \\ (\mathbf{e}_a, \mathbf{m}) \end{array} \right]_q .$$

As a corollary we have

Corollary 1.6

$$\chi_{1,r(k+1)}^{(p,kp+p-1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2+\cdots+n_k^2+(r-1)(n_1+\cdots+n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+r-1}}$$

$$\times \sum_{\mathbf{m}\in 2\mathbb{Z}^n+\mathbf{Q}_{p-1-r,p-1}} q^{(\mathbf{m},C_{p-3}\mathbf{m})/4} \tag{1.15}$$

$$\times \prod_{a=1}^{p-3} \left[\begin{array}{c} (\mathbf{e}_a, I_{p-3}\mathbf{m} + \mathbf{e}_{p-1-r} + (2n_k+r-1)\mathbf{e}_1)/2 \\ (\mathbf{e}_a, \mathbf{m}) \end{array} \right]_q .$$

$$\chi_{1,r(k+1)+k}^{(p,kp+p-1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2+\cdots+n_k^2+r(n_1+\cdots+n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+r}}$$

$$\times \sum_{\mathbf{m}\in 2\mathbb{Z}^n+\mathbf{Q}_{r,1}} q^{(\mathbf{m},C_{p-3}\mathbf{m})/4} \prod_{a=1}^{p-3} \left[\begin{array}{c} (\mathbf{e}_a, I_{p-3}\mathbf{m} + \mathbf{e}_r + (2n_k+r)\mathbf{e}_1)/2 \\ (\mathbf{e}_a, \mathbf{m}) \end{array} \right]_q . \tag{1.16}$$

Theorem 1.7

$$\frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \left(q^{p(kp+1)n^2+(kp+1-krp)n} - q^{(pn+1)((kp+1)n+kr)} \right)$$

$$= \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2+\cdots+n_{k-1}^2+2n_k^2++(r-1)(n_1+\cdots+n_{k-1}+2n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+r-1}}$$

$$\times \sum_{\mathbf{m}\in 2\mathbb{Z}^n+\mathbf{Q}_{p-1-r,p-1}} q^{(\mathbf{m},C_{p-3}\mathbf{m})/4-(\mathbf{e}_{p-1-r}+(2n_k+r-1)\mathbf{e}_1,\mathbf{m})/2} \tag{1.17}$$

$$\times \prod_{a=1}^{p-3} \left[\begin{array}{c} (\mathbf{e}_a, I_{p-3}\mathbf{m} + \mathbf{e}_{p-1-r} + (2n_k+r-1)\mathbf{e}_1)/2 \\ (\mathbf{e}_a, \mathbf{m}) \end{array} \right]_q .$$

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$$\frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \left(q^{p(kp+1)n^2 + (kp+1-(kr+k+1)p)n} - q^{(pn+1)((kp+1)n+kr+k+1)} \right)$$

$$= \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_{k-1}^2 + 2n_k^2 + r(n_1 + \cdots + n_{k-1} + 2n_k)}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k} (q)_{2n_k+r}}$$

$$\times \sum_{\mathbf{m} \in 2\mathbb{Z}^n + \mathbf{Q}_{r,1}} q^{(\mathbf{m}, C_{p-3}\mathbf{m})/4 - (\mathbf{e}_r + (2n_k+r)\mathbf{e}_1, \mathbf{m})/2}$$

$$\times \prod_{a=1}^{p-3} \left[\begin{array}{c} (\mathbf{e}_a, I_{p-3}\mathbf{m} + \mathbf{e}_r + (2n_k+r)\mathbf{e}_1)/2 \\ (\mathbf{e}_a, \mathbf{m}) \end{array} \right]_q . \tag{1.18}$$

As a corollary we have

Corollary 1.8

$$\chi_{1,kr}^{(p,kp+1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_{k-1}^2 + 2n_k^2 + (r-1)(n_1 + \cdots + n_{k-1} + 2n_k)}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k} (q)_{2n_k+r-1}}$$

$$\times \sum_{\mathbf{m} \in 2\mathbb{Z}^n + \mathbf{Q}_{p-1-r,p-1}} q^{(\mathbf{m}, C_{p-3}\mathbf{m})/4 - (\mathbf{e}_{p-1-r} + (2n_k+r-1)\mathbf{e}_1, \mathbf{m})/2}$$

$$\times \prod_{a=1}^{p-3} \left[\begin{array}{c} (\mathbf{e}_a, I_{p-3}\mathbf{m} + \mathbf{e}_{p-1-r} + (2n_k+r-1)\mathbf{e}_1)/2 \\ (\mathbf{e}_a, \mathbf{m}) \end{array} \right]_q . \tag{1.19}$$

$$\chi_{1,k(r+1)+1}^{(p,kp+1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_{k-1}^2 + 2n_k^2 + r(n_1 + \cdots + n_{k-1} + 2n_k)}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k} (q)_{2n_k+r}}$$

$$\times \sum_{\mathbf{m} \in 2\mathbb{Z}^n + \mathbf{Q}_{r,1}} q^{(\mathbf{m}, C_{p-3}\mathbf{m})/4 - (\mathbf{e}_r + (2n_k+r)\mathbf{e}_1, \mathbf{m})/2}$$

$$\times \prod_{a=1}^{p-3} \left[\begin{array}{c} (\mathbf{e}_a, I_{p-3}\mathbf{m} + \mathbf{e}_r + (2n_k+r)\mathbf{e}_1)/2 \\ (\mathbf{e}_a, \mathbf{m}) \end{array} \right]_q . \tag{1.20}$$

Setting $r = 1$ and $n_i - n_{i+1} = \nu_i$ $(i = 1, \cdots, k-1)$, $2n_k = \nu_k$, $m_a = \nu_{k+a}$ $(a = 1, \cdots, p-3)$ in (1.19), we reproduce the corresponding expressions given in [1].

2   The Andrews–Bailey construction

This section is devoted to a review of a number of definitions and propositions from the pioneering work of Bailey [10] and its extensions by Andrews [11].

2.1   A Bailey pair

Definition 2.1 *Let* $\alpha = \{\alpha_n(a,q)\}_{n \geq 0}$ *and* $\beta = \{\beta_n(a,q)\}_{n \geq 0}$ *be sequences of functions in a and q. They form a Bailey pair relative to a, if they satisfy the relation*

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}. \tag{2.1}$$

Note that (2.1) has the form $\beta = M\alpha$, where $\alpha = {}^t(\alpha_0, \alpha_1, \alpha_2, \cdots), \beta = {}^t(\beta_0, \beta_1, \beta_2, \cdots)$; and that M is an invertible matrix of infinite size, because it is a lower triangular matrix with non-zero diagonal elements. Using an identity of q-hypergeometric series [19, eq.(1.4.3)], one can prove the following proposition [11]

Proposition 2.1 *A pair (α, β) is a Bailey pair relative to a if and only if*

$$\alpha_n = \frac{1 - aq^{2n}}{1 - a} \sum_{r=0}^{n} \frac{(a)_{n+r}(-1)^{n-r} q^{\binom{n-r}{2}}}{(q)_{n-r}} \beta_r. \tag{2.2}$$

Note that the RHS of (2.2) has no singularity at $a = 1$.

2.2 A dual Bailey pair

Given a Bailey pair, Andrews [11] proposed a method to construct a new Bailey pair as follows:

Proposition 2.2 *If $\alpha = \{\alpha_n(a, q)\}_{n \geq 0}$ and $\beta = \{\beta_n(a, q)\}_{n \geq 0}$ form a Bailey pair, Then the sequences $A = \{A_n(a, q)\}_{n \geq 0}$ and $B = \{B_n(a, q)\}_{n \geq 0}$ defined by*

$$\begin{aligned} A_n(a, q) &= a^n q^{n^2} \alpha_n(a^{-1}, q^{-1}), \\ B_n(a, q) &= a^{-n} q^{-n^2-n} \beta_n(a^{-1}, q^{-1}), \end{aligned} \tag{2.3}$$

form another Bailey pair relative to a.

One can prove Proposition (2.2) by substituting $\alpha_n(a^{-1}, q^{-1}), \beta_n(a^{-1}, q^{-1})$ into (2.1). We refer to (A, B) as the dual of (α, β) and *vice versa*.

2.3 A Bailey chain

Using Saalschütz theorem [19, eq.(1.7.2)], one can prove the celebrated Bailey's lemma [10]:

Lemma 2.3 *Let α and β be a Bailey pair relative to a, and set*

$$\alpha'_n = a^n q^{n^2} \alpha_n, \quad \beta'_n = \sum_{r=0}^{n} \frac{a^r q^{r^2}}{(q)_{n-r}} \beta_r. \tag{2.4}$$

Then α' and β' is also a Bailey pair relative to a.

Andrews [11] proposed a prescription to construct an infinite sequences of Bailey pairs starting from a given pair using Bailey's lemma. Given a Bailey pair $(\alpha^{(0)}, \beta^{(0)})$, one can obtain another Bailey pair $(\alpha^{(1)}, \beta^{(1)})$ by applying (2.4). Repeating this procedure, one obtains $(\alpha^{(k)}, \beta^{(k)})$ inductively from $(\alpha^{(k-1)}, \beta^{(k-1)})$. The resulting infinite sequence of pairs is called a *Bailey chain*.

Substituting $\alpha^{(k)}$ and $\beta^{(k)}$ into the defining relation (2.1) to get

$$\sum_{r=0}^{n} \frac{a^{kr} q^{kr^2} \alpha_r^{(0)}}{(q)_{n-r}(aq)_{n+r}} = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n_1} \cdots \sum_{n_k=0}^{n_{k-1}} \frac{a^{n_1+\cdots n_k} q^{n_1^2+\cdots+n_k^2} \beta_{n_k}^{(0)}}{(q)_{n-n_1}(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k}},$$

and taking the limit $n \to \infty$, Andrews [11] obtained the following identity:

Corollary 2.4 *Let (α, β) be a Bailey pair relative to a. The following holds:*

$$\frac{1}{(aq)_\infty} \sum_{n=0}^\infty a^{kn} q^{kn^2} \alpha_n = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{a^{n_1 + \cdots n_k} q^{n_1^2 + \cdots + n_k^2} \beta_{n_k}}{(q)_{n_1 - n_2} \cdots (q)_{n_{k-1} - n_k}}. \tag{2.5}$$

The following remark is the central to this work: for certain α_n, we will be able to rewrite the LHS of (2.5) in the form of (1.1), thus obtaining a q-series identity of the type we wish to prove.

2.4 A Bailey lattice

In a Bailey chain, the parameter a remains constant throughout the chain. In [12, 13], yet another extension of Bailey's lemma was presented, that allows one to vary a. The resulting structure is called a *Bailey lattice*. Using q-hypergeometric series identities [19, eqs.(1.7.2),(2.2.4) and (2.3.4)] one can prove the following Proposition [12, 13]:

Proposition 2.5 *Let α and β be a Bailey pair relative to a and set*

$$\alpha_n' = \begin{cases} \alpha_0, & n = 0, \\ (1-a)a^n q^{n^2-n} \left\{ \dfrac{\alpha_n}{1 - aq^{2n}} - \dfrac{aq^{2n-2}\alpha_{n-1}}{1 - aq^{2n-2}} \right\}, & n > 0, \end{cases}$$

$$\beta_n' = \sum_{r=0}^n \frac{a^r q^{r^2-r}}{(q)_{n-r}} \beta_r. \tag{2.6}$$

Then α' and β' is also a Bailey pair relative to aq^{-1}.

Using the above extension, we can construct a new Bailey pair $(\alpha^{(k)}(aq^{-1}, q), \beta^{(k)}(aq^{-1}, q))$ from a given pair $(\alpha^{(0)}(a, q), \beta^{(0)}(a, q))$ as follows. In the first $k - i - 1$ steps, we transform $(\alpha^{(0)}(a, q), \beta^{(0)}(a, q))$ by (2.4) as before. At the $(k - i)^{\text{th}}$ step, we use (2.6). After that, we transform $(\alpha^{(k-i)}(aq^{-1}, q), \beta^{(k-i)}(aq^{-1}, q))$ by (2.4) i times. Substituting $(\alpha^{(k)}(aq^{-1}, q), \beta^{(k)}(aq^{-1}, q))$ into the defining relation (2.1), and taking the limit $n \to \infty$, we obtain the following Corollary [12].

Corollary 2.6 *Let (α, β) be a Bailey pair. Then the following identity holds:*

$$\sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{a^{n_1 + \cdots n_k} q^{n_1^2 + \cdots + n_k^2 - n_1 - \cdots - n_i} \beta_{n_k}}{(q)_{n_1 - n_2} \cdots (q)_{n_{k-1} - n_k}}$$

$$= \frac{1}{(a)_\infty} \left\{ \alpha_0 + (1 - a) \sum_{n=1}^\infty \left[\frac{a^{kn} q^{kn^2 - in} \alpha_n}{1 - aq^{2n}} - \frac{a^{k(n-1)} q^{k(n-1)^2 - i(n-1)} (aq^{2(n-1)})^{i+1} \alpha_{n-1}}{1 - aq^{2n-2}} \right] \right\}. \tag{2.7}$$

2.5 Example

As a starting point, let us choose the following pair of sequences [12]

$$\alpha_n^{(0)} = \begin{cases} 1, & n = 0, \\ (-1)^n q^{(n^2-n)/2} \dfrac{1 - aq^{2n}}{1 - a} \dfrac{(a)_n}{(q)_n}, & n > 0, \end{cases}$$

$$\beta_n^{(0)} = \delta_{n0}. \tag{2.8}$$

From Proposition 2.1 $(\alpha^{(0)}, \beta^{(0)})$ is a Bailey pair. In this example the dual Bailey pair is the original one itself.

Consider the Bailey chain $\{(\alpha^{(k)}, \beta^{(k)})\}_{k \geq 0}$, for $a = q^j (j = 0, 1)^2$. Applying Corollary 2.4 to $(\alpha^{(0)}, \beta^{(0)})$, we obtain

$$\frac{1}{(q)_\infty} \sum_{r=-\infty}^{\infty} (-1)^r q^{r((2k+1)r+1-2jk)/2} = \sum_{n_1 \geq \cdots \geq n_{k-1} \geq 0} \frac{q^{n_1^2 + \cdots + n_{k-1}^2 + j(n_1 + \cdots + n_{k-1})}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-2}-n_{k-1}} (q)_{n_{k-1}}}. \tag{2.9}$$

Setting $k = 1$, (2.9) reduces to Euler's pentagonal numbers theorem [21]

$$(q)_\infty = 1 + \sum_{r=1}^{\infty} (-1)^r (q^{r(3r-1)/2} + q^{r(3r+1)/2}); \tag{2.10}$$

and setting $k = 2$, (2.9) reduces to the Rogers–Ramanujan identity [7]

$$\frac{1}{(q)_\infty} \sum_{r=-\infty}^{\infty} (-1)^r q^{r(5r+1)/2 - 2jr} = \sum_{n=0}^{\infty} \frac{q^{n+jn}}{(q)_n}. \tag{2.11}$$

Applying Corollary 2.6 to $(\alpha^{(0)}, \beta^{(0)})$ for $a = q$ and replace i by $i - 1$, we obtain Gordon's generalization of Rogers–Ramanujan identity [7]

$$\frac{1}{(q)_\infty} \sum_{r=-\infty}^{\infty} (-1)^r q^{r((2k+1)r+2k-2i+1)/2} = \sum_{n_1 \geq \cdots \geq n_{k-1} \geq 0} \frac{q^{n_1^2 + \cdots + n_{k-1}^2 + n_i + \cdots + n_{k-1}}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-2}-n_{k-1}} (q)_{n_{k-1}}}, \tag{2.12}$$

where $1 \leq i \leq k$. By comparing with (1.4) one can see that the LHS and RHS of (2.12) are the bosonic and fermionic representations of $\chi_{1,i}^{(2,2k+1)}(q)$, respectively.

3 Fermionic sum representations of the type (II)

In this section we apply the Andrews-Bailey construction to the polynomial identity obtained in [6].

3.1 A polynomial identity

Our starting point is the following polynomial identity which implies Gordon's generalization of the Rogers–Ramanujan identities (2.12):

Proposition 3.1 *Let ν, κ, ι be fixed non negative integers such that $1 \leq \iota \leq \kappa$. Then the following polynomial identities hold.*

$$\sum_{\rho=-\infty}^{\infty} (-1)^\rho q^{\rho((2\kappa+1)\rho+2\kappa-2\iota+1)/2} \left[\begin{array}{c} \nu \\ \left[\frac{\nu-\kappa+\iota-(2\kappa+1)\rho}{2} \right] \end{array} \right]_q$$

$$= \sum_{\substack{\nu_1 \geq \cdots \geq \nu_{\kappa-1} \geq \nu_\kappa = 0 \\ 2(\nu_1 + \cdots + \nu_{\kappa-1}) \leq \nu-\kappa+\iota}} q^{\nu_1^2 + \cdots + \nu_{\kappa-1}^2 + \nu_\iota + \cdots + \nu_{\kappa-1}} \times \tag{3.1}$$

$$\times \prod_{\mu=1}^{\kappa-1} \left[\begin{array}{c} \nu - 2(\nu_1 + \cdots + \nu_{\mu-1}) - \nu_\mu - \nu_{\mu+1} - \alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu - \nu_{\mu+1} \end{array} \right]_q.$$

---

[2] When $j = 0$ (resp. $j = 1$) the pair $(\alpha^{(1)}, \beta^{(1)})$ coincides B(1) (resp. B(3)) in Slater's table [20].

Here, $[x]$ is the greatest integer part of x,

$$
\left[\begin{array}{c} N \\ M \end{array}\right]_q = \left\{ \begin{array}{ll} \dfrac{(q)_N}{(q)_M (q)_{N-M}}, & 0 \le M \le N, \\[2mm] 0, & otherwise. \end{array} \right.
$$

is a q-binomial coefficient, and

$$
\alpha_{\iota\mu}^{(\kappa)} = \left\{ \begin{array}{ll} 0, & if\ 1 \le \mu \le \iota - 1, \\[2mm] \mu - \iota + 1, & if\ \iota \le \mu \le \kappa - 1. \end{array} \right.
$$

The proof is given in [6]. For $\kappa = 1, 2$, this identity appears in [7].

3.2   Fermionic sum representations for $\mathcal{M}(2\kappa + 1, (2\kappa + 1)k + 2)$

Set $\nu = 2n + \kappa - \iota$ and divide (3.1) by $(q^{\kappa-\iota+1})_{2n}$. Then we have

$$
\sum_{r=0}^{\infty} \frac{q^{r((4\kappa+2)r+2\kappa-2\iota+1)}}{(q)_{n-(2\kappa+1)r}(q^{\kappa-\iota+1})_{n+(2\kappa+1)r}} + \sum_{r=0}^{\infty} \frac{q^{r((4\kappa+2)r-2\kappa+2\iota-1)}}{(q)_{n+\kappa-\iota-(2\kappa+1)r}(q^{\kappa-\iota+1})_{n-\kappa+\iota+(2\kappa+1)r}}
$$
$$
- \sum_{r=1}^{\infty} \frac{q^{(2r-1)((2\kappa+1)r-\iota)}}{(q)_{n+\kappa-(2\kappa+1)r}(q^{\kappa-\iota+1})_{n-\kappa+(2\kappa+1)r}} - \sum_{r=0}^{\infty} \frac{q^{(2r+1)((2\kappa+1)r+\iota)}}{(q)_{n-\iota-(2\kappa+1)r}(q^{\kappa-\iota+1})_{n+\iota+(2\kappa+1)r}}
$$
$$
= \frac{1}{(q^{\kappa-\iota+1})_{2n}} \sum_{\substack{\nu_1 \ge \cdots \ge \nu_{\kappa-1} \ge \nu_\kappa = 0 \\ \nu_1 + \cdots + \nu_{\kappa-1} \le n}} q^{\nu_1^2 + \cdots + \nu_{\kappa-1}^2 + \nu_\iota + \cdots + \nu_{\kappa-1}} \times
$$
$$
\times \prod_{\mu=1}^{\kappa-1} \left[\begin{array}{c} 2n + \kappa - \iota - 2(\nu_1 + \cdots + \nu_{\mu-1}) - \nu_\mu - \nu_{\mu+1} - \alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu - \nu_{\mu+1} \end{array}\right]_q.
$$

From the above we can read the following Bailey pair relative to $q^{\kappa-\iota}$:

$$
\alpha_n = \left\{ \begin{array}{ll} 1, & n = 0, \\[1mm] q^{r((4\kappa+2)r+2\kappa-2\iota+1)}, & n = (2\kappa+1)r\ (r \ge 1), \\[1mm] q^{r((4\kappa+2)r-2\kappa+2\iota-1)}, & n = (2\kappa+1)r - \kappa + \iota\ (r \ge 1), \\[1mm] -q^{(2r-1)((2\kappa+1)r-\iota)}, & n = (2\kappa+1)r - \kappa\ (r \ge 1), \\[1mm] -q^{(2r+1)((2\kappa+1)r+\iota)}, & n = (2\kappa+1)r + \iota\ (r \ge 0), \\[1mm] 0, & otherwise, \end{array} \right.
\tag{3.2}
$$

$$
\begin{aligned}
\beta_n &= \frac{1}{(q^{\kappa-\iota+1})_{2n}} \sum_{\substack{\nu_1 \ge \cdots \ge \nu_{\kappa-1} \ge \nu_\kappa = 0 \\ \nu_1 + \cdots + \nu_{\kappa-1} \le n}} q^{\nu_1^2 + \cdots + \nu_{\kappa-1}^2 + \nu_\iota + \cdots + \nu_{\kappa-1}} \times \\
&\quad \times \prod_{\mu=1}^{\kappa-1} \left[\begin{array}{c} 2n + \kappa - \iota - 2(\nu_1 + \cdots + \nu_{\mu-1}) - \nu_\mu - \nu_{\mu+1} - \alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu - \nu_{\mu+1} \end{array}\right]_q.
\end{aligned}
\tag{3.3}
$$

Here and hereafter terms belonging to the same congruence class modulo $2\kappa + 1$ should be summed up if exists. For instance, when $\iota = \kappa$ (3.2) should read as follows:

$$
\alpha_n = \left\{ \begin{array}{ll} 1, & n = 0, \\[1mm] q^{r((4\kappa+2)r+1)} + q^{r((4\kappa+2)r-1)}, & n = (2\kappa+1)r\ (r \ge 1), \\[1mm] -q^{(2r-1)((2\kappa+1)r-\kappa)}, & n = (2\kappa+1)r - \kappa\ (r \ge 1), \\[1mm] -q^{(2r+1)((2\kappa+1)r+\kappa)}, & n = (2\kappa+1)r + \kappa\ (r \ge 0), \\[1mm] 0, & otherwise, \end{array} \right.
\tag{3.4}
$$

Noting that

$$\sum_{n=0}^{\infty}(q^{\kappa-\iota})^{kn}q^{kn^2}\alpha_n$$

$$= 1+\sum_{r=1}^{\infty}(q^{\kappa-\iota})^{k(2\kappa+1)r}q^{k(2\kappa+1)^2r^2}q^{r(2(2\kappa+1)r+2\kappa-2\iota+1)}$$

$$+ \sum_{r=1}^{\infty}(q^{\kappa-\iota})^{k((2\kappa+1)r-\kappa+\iota)}q^{k((2\kappa+1)r-\kappa+\iota)^2}q^{r(2(2\kappa+1)r-2\kappa+2\iota+1)}$$

$$- \sum_{r=1}^{\infty}(q^{\kappa-\iota})^{k((2\kappa+1)r-\kappa)}q^{k((2\kappa+1)r-\kappa)^2}q^{(2r-1)((2\kappa+1)r-\iota)}$$

$$- \sum_{r=0}^{\infty}(q^{\kappa-\iota})^{k((2\kappa+1)r-\iota)}q^{k((2\kappa+1)r-\iota)^2}q^{(2r+1)((2\kappa+1)r+\iota)}$$

$$= \sum_{r=-\infty}^{\infty}\left(q^{r((2\kappa+1)((2\kappa+1)k+2)r+(2\kappa+1)(\kappa-\iota+1)-2\iota)}-q^{((2\kappa+1)r+\iota)(((2\kappa+1)k+2)r+k\kappa+1)}\right)$$

$$= (q)_{\infty}\chi_{\iota,k\kappa+1}^{(2\kappa+1,(2\kappa+1)k+2)}(q),$$

and applying Corollary 2.4 to the Bailey pair (3.2–3.3), we obtain the following fermionic expression for the Virasoro character:

$$\chi_{\iota,k\kappa+1}^{(2\kappa+1,(2\kappa+1)k+2)}(q) = \sum_{n_1\geq\cdots\geq n_k\geq 0}\frac{q^{n_1^2+\cdots+n_k^2+(\kappa-\iota)(n_1+\cdots+n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+\kappa-\iota}}$$
$$\times \sum_{\substack{\nu_1\geq\cdots\geq\nu_{\kappa-1}\geq\nu_\kappa=0 \\ \nu_1+\cdots+\nu_{\kappa-1}\leq n_k}}q^{\nu_1^2+\cdots+\nu_{\kappa-1}^2+\nu_\iota+\cdots+\nu_{\kappa-1}}$$
$$\times \prod_{\mu=1}^{\kappa-1}\left[\begin{array}{c}2n_k+\kappa-\iota-2(\nu_1+\cdots+\nu_{\mu-1})-\nu_\mu-\nu_{\mu+1}-\alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu-\nu_{\mu+1}\end{array}\right]_q. \tag{3.5}$$

Setting $k=1, \iota=\kappa=(p-1)/2$ and $2n_1=m_1, 2\nu_1=m_1-m_2,\cdots,2\nu_{\kappa-1}=m_{\kappa-1}-m_\kappa$ in (3.5), we reproduce the corresponding expressions in [1].

Substituting $\nu=2n+\kappa-\iota+1$ into (3.1) and dividing by $(q^{\kappa-\iota+2})_{2n}$ we obtain another Bailey pair relative to $q^{\kappa-\iota+1}$

$$\alpha_n = \begin{cases} 1, & n=0, \\ q^{r((4\kappa+2)r+2\kappa-2\iota+1)}, & n=(2\kappa+1)r\ (r\geq 1), \\ q^{r((4\kappa+2)r-2\kappa+2\iota-1)}, & n=(2\kappa+1)r-\kappa+\iota-1\ (r\geq 1), \\ -q^{(2r+1)((2\kappa+1)r+\iota)}, & n=(2\kappa+1)r+\iota\ (r\geq 0), \\ -q^{(2r-1)((2\kappa+1)r-\iota)}, & n=(2\kappa+1)r-\kappa-1\ (r\geq 1), \\ 0, & \text{otherwise,} \end{cases} \tag{3.6}$$

$$\beta_n = \frac{1}{(q^{\kappa-\iota+2})_{2m}}\sum_{\substack{\nu_1\geq\cdots\geq\nu_{\kappa-1}\geq\nu_\kappa=0 \\ \nu_1+\cdots+\nu_{\kappa-1}\leq n}}q^{\nu_1^2+\cdots+\nu_{\kappa-1}^2+\nu_\iota+\cdots+\nu_{\kappa-1}}$$
$$\times \prod_{\mu=1}^{\kappa-1}\left[\begin{array}{c}2n+\kappa-\iota+1-2(\nu_1+\cdots+\nu_{\mu-1})-\nu_\mu-\nu_{\mu+1}-\alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu-\nu_{\mu+1}\end{array}\right]_q. \tag{3.7}$$

Applying Corollary 2.4 to this Bailey pair we obtain the following fermionic expression for the Virasoro

character:

$$
\begin{aligned}
\chi_{\iota,k(\kappa+1)+1}^{(2\kappa+1,(2\kappa+1)k+2)}(q) \quad = \quad & \sum_{n_1\geq\cdots\geq n_k\geq 0} \frac{q^{n_1^2\cdots+n_k^2+(\kappa-\iota+1)(n_1+\cdots+n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+\kappa-\iota+1}} \\
\times \quad & \sum_{\substack{\nu_1\geq\cdots\geq\nu_{\kappa-1}\geq\nu_\kappa=0 \\ \nu_1+\cdots+\nu_{\kappa-1}\leq n_k}} q^{\nu_1^2+\cdots+\nu_{\kappa-1}^2+\nu_\iota+\cdots+\nu_{\kappa-1}} \\
\times \quad & \prod_{\mu=1}^{\kappa-1} \left[\begin{array}{c} 2n_k+\kappa-\iota+1-2(\nu_1+\cdots+\nu_{\mu-1})-\nu_\mu-\nu_{\mu+1}-\alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu-\nu_{\mu+1} \end{array} \right]_q .
\end{aligned}
\tag{3.8}
$$

Furthermore, applying Corollary 2.6 to this pair (3.6–3.7) we get

$$
\begin{aligned}
& \sum_{j=0}^{i} q^{j(\kappa-\iota+1)}\chi_{\iota,(\kappa+1)k+1-i+2j}^{(2\kappa+1,(2\kappa+1)k+2)}(q) \\
= \quad & \sum_{n_1\geq\cdots\geq n_k\geq 0} \frac{q^{n_1^2\cdots+n_k^2+(\kappa-\iota)(n_1+\cdots+n_i)+(\kappa-\iota+1)(n_{i+1}+\cdots+n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+\kappa-\iota+1}} \\
\times \quad & \sum_{\substack{\nu_1\geq\cdots\geq\nu_{\kappa-1}\geq\nu_\kappa=0 \\ \nu_1+\cdots+\nu_{\kappa-1}\leq n_k}} q^{\nu_1^2+\cdots+\nu_{\kappa-1}^2+\nu_\iota+\cdots+\nu_{\kappa-1}} \\
\times \quad & \prod_{\mu=1}^{\kappa-1} \left[\begin{array}{c} 2n_k+\kappa-\iota+1-2(\nu_1+\cdots+\nu_{\mu-1})-\nu_\mu-\nu_{\mu+1}-\alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu-\nu_{\mu+1} \end{array} \right]_q .
\end{aligned}
\tag{3.9}
$$

where $0 \leq i \leq k$. Notice that the LHS of the above equation is a weighted sum of Virasoro characters. Extracting single character from such an equation is beyond the scope of this work.

3.3 Fermionic sum representations for $\mathcal{M}(2\kappa+1,(2\kappa+1)k+2\kappa-1)$

The dual Bailey pair to (3.2–3.3) is

$$
A_n = \begin{cases}
1, & n=0, \\
q^{r((2\kappa+1)(2\kappa-1)r+(2\kappa+1)(\kappa-\iota-1)+2\iota)}, & n=(2\kappa+1)r \ (r\geq 1), \\
q^{r((2\kappa+1)(2\kappa-1)r-(2\kappa+1)(\kappa-\iota-1)+2\iota)}, & n=(2\kappa+1)r-\kappa+\iota, \ (r\geq 1) \\
-q^{((2\kappa+1)r-\iota)((2\kappa-1)r-\kappa+1)}, & n=(2\kappa+1)r-\kappa \ (r\geq 1), \\
-q^{((2\kappa+1)r+\iota)((2\kappa-1)r+\kappa-1)}, & n=(2\kappa+1)r+\iota \ (r\geq 0), \\
0, & \text{otherwise,}
\end{cases}
\tag{3.10}
$$

$$
\begin{aligned}
B_n \quad = \quad & \frac{q^{n^2+(\kappa-\iota)n}}{(q^{\kappa-\iota+1})_{2n}} \sum_{\substack{\nu_1\geq\cdots\geq\nu_{l-1}\geq\nu_\kappa=0 \\ \nu_1+\cdots+\nu_{\kappa-1}\leq n}} q^{\nu_1^2+\cdots+\nu_{\kappa-1}^2-\nu_1(2m+\kappa-\iota)}\times \\
\times \quad & \prod_{\mu=1}^{\kappa-1} \left[\begin{array}{c} 2n+\kappa-\iota-2(\nu_1+\cdots+\nu_{\mu-1})-\nu_\mu-\nu_{\mu+1}-\alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu-\nu_{\mu+1} \end{array} \right]_q .
\end{aligned}
\tag{3.11}
$$

In order to obtain B_m, we use

$$
\left[\begin{array}{c} N+M \\ M \end{array} \right]_{q^{-1}} = q^{-NM} \left[\begin{array}{c} N+M \\ M \end{array} \right]_q ,
$$

and

$$\sum_{\mu=1}^{\kappa-1}(\nu_\mu - \nu_{\mu+1})(2(\nu_1 + \cdots + \nu_\mu) + \alpha_{\iota\mu}^{(\kappa)}) = 2\sum_{\mu=1}^{\kappa-1}\nu_\mu^2 + \sum_{\mu=\iota}^{\kappa-1}\nu_\mu.$$

From

$$\frac{1}{(q)_\infty}\sum_{n=0}^{\infty}A_n(q^{\kappa-\iota})^{kn}q^{kn^2} = \chi_{\iota,\kappa k+\kappa-1}^{(2\kappa+1,(2\kappa+1)k+2\kappa-1)}(q), \tag{3.12}$$

and Corollary 2.4, we obtain

$$
\begin{aligned}
\chi_{\iota,\kappa k+\kappa-1}^{(2\kappa+1,(2\kappa+1)(k+1)-2)}(q) \quad &= \quad \sum_{n_1\geq\cdots\geq n_k\geq 0}\frac{q^{n_1^2+\cdots+n_{k-1}^2+2n_k^2+(\kappa-\iota)(n_1+\cdots+n_{k-1}+2n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+\kappa-\iota}} \\
&\times \sum_{\substack{\nu_1\geq\cdots\geq\nu_{\kappa-1}\geq\nu_\kappa=0 \\ \nu_1+\cdots+\nu_{\kappa-1}\leq n_k}} q^{\nu_1^2+\cdots+\nu_{\kappa-1}^2-\nu_1(2n_k+\kappa-\iota)}\times \\
&\times \prod_{\mu=1}^{\kappa-1}\left[\begin{array}{c} 2n_k+\kappa-\iota-2(\nu_1+\cdots+\nu_{\mu-1})-\nu_\mu-\nu_{\mu+1}-\alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu-\nu_{\mu+1} \end{array}\right]_q.
\end{aligned}
\tag{3.13}
$$

The dual Bailey pair to (3.6–3.7) is

$$
A_n = \begin{cases}
1, & n=0, \\
q^{r((2\kappa+1)(2\kappa-1)r+(2\kappa+1)(\kappa-\iota)+2\iota)}, & n=(2\kappa+1)r\ (r\geq 1), \\
q^{r((2\kappa+1)(2\kappa-1)r-(2\kappa+1)(\kappa-\iota)+2\iota)}, & n=(2\kappa+1)r-\kappa+\iota-1\ (r\geq 1), \\
-q^{((2\kappa-1)r+\kappa)((2\kappa+1)r+\iota)}, & n=(2\kappa+1)r+\iota\ (r\geq 0), \\
-q^{((2\kappa-1)r-\kappa)((2\kappa+1)r-\iota)}, & n=(2\kappa+1)r-\kappa-1\ (r\geq 1), \\
0, & \text{otherwise,}
\end{cases}
\tag{3.14}
$$

$$
\begin{aligned}
B_n \quad &= \quad \frac{q^{n^2+(\kappa-\iota+1)n}}{(q^{\kappa-\iota+2})_{2n}}\sum_{\substack{\nu_1\geq\cdots\geq\nu_{\kappa-1}\geq\nu_\kappa=0 \\ \nu_1+\cdots+\nu_{\kappa-1}\leq n}} q^{\nu_1^2+\cdots+\nu_{\kappa-1}^2-\nu_1(2n+\kappa-\iota+1)} \\
&\times \prod_{\mu=1}^{\kappa-1}\left[\begin{array}{c} 2n+\kappa-\iota+1-2(\nu_1+\cdots+\nu_{\mu-1})-\nu_\mu-\nu_{\mu+1}-\alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu-\nu_{\mu+1} \end{array}\right]_q.
\end{aligned}
\tag{3.15}
$$

Applying Corollary 2.4 to this Bailey pair we obtain the following fermionic expression for the Virasoro character:

$$
\begin{aligned}
\chi_{\iota,k(\kappa+1)+\kappa}^{(2\kappa+1,(2\kappa+1)k+2\kappa-1)}(q) \quad &= \quad \sum_{n_1\geq\cdots\geq n_k\geq 0}\frac{q^{n_1^2\cdots+n_{k-1}^2+2n_k^2+(\kappa-\iota+1)(n_1+\cdots+n_{k-1}+2n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+\kappa-\iota+1}} \\
&\times \sum_{\substack{\nu_1\geq\cdots\geq\nu_{\kappa-1}\geq\nu_\kappa=0 \\ \nu_1+\cdots+\nu_{\kappa-1}\leq n_k}} q^{\nu_1^2+\cdots+\nu_{\kappa-1}^2-\nu_1(\kappa-\iota+1)} \\
&\times \prod_{\mu=1}^{\kappa-1}\left[\begin{array}{c} 2n_k+\kappa-\iota+1-2(\nu_1+\cdots+\nu_{\mu-1})-\nu_\mu-\nu_{\mu+1}-\alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu-\nu_{\mu+1} \end{array}\right]_q.
\end{aligned}
\tag{3.16}
$$

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Furthermore, applying Corollary 2.6 to this pair (3.14–3.15) we get

$$
\begin{aligned}
&\sum_{j=0}^{i} q^{j(\kappa-\iota+1)} \chi_{\iota,(\kappa+1)k+\kappa-i+2j}^{(2\kappa+1,(2\kappa+1)k+2\kappa-1)}(q) \\
&= \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 \cdots + n_{k-1}^2 + 2n_k^2 + +(\kappa-\iota+1)(n_1+\cdots+n_{k-1}+2n_k)-(n_1+\cdots+n_i)}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k} (q)_{2n_k+\kappa-\iota+1}} \\
&\times \sum_{\substack{\nu_1 \geq \cdots \geq \nu_{\kappa-1} \geq \nu_\kappa=0 \\ \nu_1+\cdots+\nu_{\kappa-1} \leq n_k}} q^{n_1^2+\cdots+n_{\kappa-1}^2+\nu_1(2n_k+\kappa-\iota+1)} \\
&\times \prod_{\mu=1}^{\kappa-1} \left[\begin{array}{c} 2n_k + \kappa - \iota + 1 - 2(\nu_1 + \cdots + \nu_{\mu-1}) - \nu_\mu - \nu_{\mu+1} - \alpha_{\iota\mu}^{(\kappa)} \\ \nu_\mu - \nu_{\mu+1} \end{array} \right]_q.
\end{aligned}
\tag{3.17}
$$

where $0 \leq i \leq k$.

4 Fermionic sum representations of the type (III)

In this section we apply the Andrews–Bailey construction to the polynomial identity obtained in [4, 5].

4.1 Another polynomial identity

Let I_n and $C_n = 2 - I_n$ stand for the incidence and Cartan matrices of the Lie algebra A_n, respectively:

$$
(I_n)_{ab} = \delta_{a,b+1} + \delta_{a,b-1} \quad a, b = 1, \cdots, n.
$$

Let $\mathbf{e}_a$ be the ath unit vector in $\mathbb{C}^n$ and set $\mathbf{e}_a = \mathbf{0}$ for $a < 0$ or $a > n$. Define the following symbol

$$
F_n^{(L)} \left[\begin{array}{c} \mathbf{Q} \\ \mathbf{A} \end{array} \right] (\mathbf{u}|q) = \sum_{\mathbf{m} \in 2\mathbb{Z}^n + \mathbf{Q}} q^{(\mathbf{m},C_n\mathbf{m})/4-(\mathbf{A},\mathbf{m})/2} \prod_{a=1}^{n} \left[\begin{array}{c} (\mathbf{e}_a, I_n\mathbf{m} + \mathbf{u} + L\mathbf{e}_1)/2 \\ (\mathbf{e}_a, \mathbf{m}) \end{array} \right]_q,
\tag{4.1}
$$

where $(\mathbf{x}, \mathbf{y})$ is the standard inner product in $\mathbb{C}^n$, and

$$
\left[\begin{array}{c} N \\ M \end{array} \right]_q = \left\{ \begin{array}{ll} \dfrac{(q)_N}{(q)_M (q)_{N-M}}, & \text{if } 0 \leq M \leq N, \\ 0, & \text{otherwise.} \end{array} \right.
$$

For $n = p - 2 \geq 1$, set

$$
\mathbf{Q}_{r,s} = (s-1)(\mathbf{e}_1 + \cdots + \mathbf{e}_n) + (\mathbf{e}_{r-1} + \mathbf{e}_{r-3} + \cdots) + (\mathbf{e}_{p+1-s} + \mathbf{e}_{p+3-s} + \cdots).
$$

and define the following two q-series:

$$
F_{p;r}^{(L)}(q) = q^{-r(r-1)} \times \left\{ \begin{array}{ll} F_{p-2}^{(L)} \left[\begin{array}{c} \mathbf{Q}_{r,1} \\ \mathbf{0} \end{array} \right] (\mathbf{e}_r|q) & \text{if } L \not\equiv r - 1 \bmod 2, \\ F_{p-2}^{(L)} \left[\begin{array}{c} \mathbf{Q}_{p-r,p} \\ \mathbf{0} \end{array} \right] (\mathbf{e}_{p-r}|q) & \text{if } L \equiv r - 1 \bmod 2, \end{array} \right.
\tag{4.2}
$$

$$\overline{F}_{p;r}^{(L)}(q) = q^{-r(r-1)} \times \begin{cases} F_{p-2}^{(L)} \begin{bmatrix} \mathbf{Q}_{r,1} \\ \mathbf{e}_r + L\mathbf{e}_1 \end{bmatrix} (\mathbf{e}_r|q) & \text{if } L \not\equiv r-1 \bmod 2, \\ F_{p-2}^{(L)} \begin{bmatrix} \mathbf{Q}_{p-r,p} \\ \mathbf{e}_{p-r} + L\mathbf{e}_1 \end{bmatrix} (\mathbf{e}_{p-r}|q) & \text{if } L \equiv r-1 \bmod 2. \end{cases} \tag{4.3}$$

We also introduce

$$\begin{aligned} B_{p;r}^{(L)}(q) &= \sum_{j\in\mathbb{Z}} \left(q^{j(jp(p+1)+r(p+1)-p)} \begin{bmatrix} L \\ \left[\frac{L-r+1}{2}\right] - j(p+1) \end{bmatrix}_q \right. \\ &- \left. q^{(jp+r)(j(p+1)+1)} \begin{bmatrix} L \\ \left[\frac{L-r-1}{2}\right] - j(p+1) \end{bmatrix}_q \right). \end{aligned} \tag{4.4}$$

Proposition 4.1 *The following polynomial identity holds:*

$$B_{p;r}^{(L)}(q) = F_{p;r}^{(L)}(q). \tag{4.5}$$

As for the proof see [5].

4.2 Fermionic sum representations for $\mathcal{M}(p, kp + p - 1)$

In what follows we replace p by $p - 1$. Accordingly, $p \geq 4$ and $1 \leq r \leq p - 2$. Set $L = 2l + r - 1$ and divide (4.5) by $(q^r)_{2l}$. Then we can read the following Bailey pair relative to q^{r-1}:

$$\alpha_l = \begin{cases} 1, & l = 0, \\ q^{j(jp(p-1)+rp-p+1)} & l = jp \ (j \geq 1), \\ q^{j(jp(p-1)-rp+p-1)} & l = jp - r + 1 \ (j \geq 1), \\ -q^{(j(p-1)+r)(jp+1)} & l = jp + 1 \ (j \geq 0), \\ -q^{(j(p-1)-r)(jp-1)} & l = jp - r \ (j \geq 1), \\ 0, & \text{otherwise}, \end{cases} \tag{4.6}$$

$$\beta_l = \frac{1}{(q^r)_{2l}} F_{p-1;r}^{(2l+r-1)}(q). \tag{4.7}$$

Here and hereafter terms belonging to the same congruence class modulo p should be summed up if exists. For instance, when $r = 1$ (4.6) should read as follows:

$$\alpha_l = \begin{cases} 1, & l = 0, \\ q^{j(jp(p-1)+rp-p+1)} + q^{j(jp(p-1)-rp+p-1)} & l = jp \ (j \geq 1), \\ -q^{(j(p-1)+r)(jp+1)} & l = jp + 1 \ (j \geq 0), \\ -q^{(j(p-1)-r)(jp-1)} & l = jp - 1 \ (j \geq 1), \\ 0, & \text{otherwise}, \end{cases} \tag{4.8}$$

By applying Lemma 2.4 to the Bailey pair (4.6–4.7), we obtain the following fermionic expression for the Virasoro character:

$$\chi_{1,r(k+1)}^{(p,kp+p-1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + (r-1)(n_1 + \cdots + n_k)}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k} (q)_{2n_k+r-1}} F_{p-1;r}^{(2n_k+r-1)}(q). \tag{4.9}$$

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Substituting $L = 2l + r$ into (4.5) and dividing by $(q^{r+1})_{2l}$ we obtain another Bailey pair relative to q^r

$$
\alpha_l = \begin{cases}
1, & l = 0, \\
q^{j(jp(p-1)+rp-p+1)} & l = jp \ (j \geq 1), \\
q^{j(jp(p-1)-rp+p-1)} & l = jp - r \ (j \geq 1), \\
-q^{(j(p-1)+r)(jp+1)} & l = jp + 1 \ (j \geq 0), \\
-q^{(j(p-1)-r)(jp-1)} & l = jp - r - 1 \ (j \geq 1), \\
0, & \text{otherwise,}
\end{cases}
\tag{4.10}
$$

$$
\beta_l = \frac{1}{(q^{r+1})_{2l}} F_{p-1;r}^{(2l+r)}(q).
\tag{4.11}
$$

Applying Lemma 2.4 to this Bailey pair we obtain the following fermionic expression for the Virasoro character:

$$
\chi_{1,r(k+1)+k}^{(p,kp+p-1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + r(n_1 + \cdots + n_k)}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k}(q)_{2n_k+r}} F_{p-1;r}^{(2n_k+r)}(q).
\tag{4.12}
$$

Furthermore, applying Corollary 2.6 to this pair (4.10–4.11) we get

$$
\sum_{j=0}^{i} q^{jr} \chi_{1,r(k+1)+k-i+2j}^{(p,kp+p-1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + (r-1)(n_1 + \cdots + n_i) + r(n_{i+1} + \cdots + n_k)}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k}(q)_{2n_k+r}} F_{p-1;r}^{(2n_k+r)}(q).
\tag{4.13}
$$

4.3 Fermionic sum representations for $\mathcal{M}(k, kp+1)$

The dual Bailey pair to (4.6–4.7) is

$$
A_l = \begin{cases}
1, & n = 0, \\
q^{j(jp-1)} & l = jp \ (j \geq 1), \\
q^{j(jp+1)} & l = jp - r + 1 \ (j \geq 1), \\
-q^{j(jp+1)} & l = jp + 1 \ (j \geq 0), \\
-q^{j(jp-1)} & l = jp - r \ (j \geq 1), \\
0 & \text{otherwise,}
\end{cases}
\tag{4.14}
$$

$$
B_l = \frac{q^{l^2 + (r-1)l}}{(q^r)_{2l}} \overline{F}_{p-1;r}^{(2l+r-1)}(q).
\tag{4.15}
$$

From

$$
\frac{1}{(q)_\infty} \sum_{l=0}^{\infty} A_l q^{(r-1)kl} q^{kl^2} = \chi_{1,kr}^{(p,kp+1)}(q),
\tag{4.16}
$$

and Lemma 2.4, we obtain

$$
\chi_{1,kr}^{(p,kp+1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_{k-1}^2 + 2n_k^2 + (r-1)(n_1 + \cdots + n_{k-1} + 2n_k)}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k}(q)_{2n_k+r-1}} \overline{F}_{p-1;r}^{(2n_k+r-1)}(q).
\tag{4.17}
$$

Setting $r = 1$ and $n_i - n_{i+1} = \nu_i \ (i = 1, \cdots, k-1)$, $2n_k = \nu_k$, $m_a = \nu_{k+a} \ (a = 1, \cdots, p-3)$ in (4.17), we reproduce the corresponding expressions given in [1].

The dual Bailey pair to (4.10–4.11) is

$$A_l = \begin{cases} 1, & n = 0, \\ q^{j(jp+p-1)} & l = jp \ (j \geq 1), \\ q^{j(jp-p+1)} & l = jp - r \ (j \geq 1), \\ -q^{(j+1)(jp+1)} & l = jp + 1 \ (j \geq 0), \\ -q^{(j-1)(jp-1)} & l = jp - r \ (j \geq 1), \\ 0, & \text{otherwise}, \end{cases} \tag{4.18}$$

$$B_l = \frac{q^{l^2+rl}}{(q^{r+1})_{2l}} \overline{F}_{p-1;r}^{(2l+r)}(q). \tag{4.19}$$

Applying Lemma 2.4 to this Bailey pair we obtain the following fermionic expression for the Virasoro character:

$$\chi_{1,k(r+1)+1}^{(p,kp+1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2+\cdots+n_{k-1}^2+2n_k^2+r(n_1+\cdots+n_{k-1}+2n_k)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+r}} \overline{F}_{p-1;r}^{(2n_k+r)}(q). \tag{4.20}$$

Furthermore, applying Corollary 2.6 to this pair (4.18–4.19) we get

$$\sum_{j=0}^{i} q^{jr} \chi_{1,k(r+1)+1-i+2j}^{(p,kp+1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2+\cdots+n_{k-1}^2+2n_k^2+r(n_1+\cdots+n_{k-1}+2n_k)-(n_1+\cdots+n_i)}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+r}} \overline{F}_{p-1;r}^{(2n_k+r)}(q). \tag{4.21}$$

5 Concluding remarks

In section 3 we derived the fermionic sum representations for Virasoro characters listed in (II) in section 1. In section 4, we derived those listed in (III). The point of this paper is to point out that the Andrews–Bailey construction provides a systematic way to prove infinite families of Viarasoro character identities. For instance, in order to derive q-series identitiies related to (II) and (III), we applied the Andrews–Bailey construction to polynomial identities obtained in [6] and [4, 5], respectively. The starting point is always a suitable Bailey pair, or equivalently a polynomial identity. Once the latter is established, the proof of an infinite family of Virasoro character identities becomes a matter of straightforward computation. In [6, Theorem 1.2] we proved another polynomial identity. Unfortunately, however, the application of the Andrews–Bailey construction to this polynomial identity dose not provide us any fruitful results.

We would like to mention fermionic representations of the type (III). In the last section we restricted ourselves $p \geq 4$. When $p = 2$, the equivalence between bosonic and fermionic representations are nothing but Gordon's generalization of the Rogers–Ramanujan identities [7]. Interpretation of Gordon's identity in terms of that equivalence was presented in [8, 9]. The case $p = 3$ of (III) is rather realized by setting $\iota = \kappa = 1$ in (II).

Proving (i) for any p, r and s is a still open problem [3]. When this can be solved, the list (III) will be extended to

$$(\widehat{III}) \quad \begin{cases} \chi_{s,r(k+1)}^{(p,kp+p-1)}(q) \text{ and } \chi_{s,r(k+1)+k}^{(p,kp+p-1)}(q) \\ \chi_{s,kr}^{(p,kp+1)}(q) \text{ and } \chi_{s,k(r+1)+1}^{(p,kp+1)}(q) \end{cases} \quad \text{for } p \geq 4, \ 1 \leq r \leq p-2, 1 \leq s \leq p-1 \text{ and } k \geq 1.$$

Furthermore, there exist a number of curious phenomena reported in [2, 15, 16, 17, 18]. We wish to discuss these matters in a separate paper.

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[3] Berkovich has recently succeeded in proving (i) for any p, r and s [22].

Acknowledgment

A   Virasoro characters and Bailey pairs

In this appendix, we list several fermionic representations obtained from Slater's table [20]. Here even when p and p' are not coprime, we denote the infinite sum defined by the RHS of (1.1) by $\chi_{r,s}^{(p,p')}(q)$.

Let us begin by considering the B and E series in Slater's table [20]. In each case $\alpha_0 = 1$.

	a	α_n	β_n
B(1)	1	$(-1)^n q^{3n^2/2}(q^{n/2} + q^{-n/2})$	$1/(q)_n$
B(2)	1	$(-1)^n q^{3n^2/2}(q^{3n/2} + q^{-3n/2})$	$q^n/(q)_n$
B(3)	q	$(-1)^n q^{(3n^2+n)/2}(1 - q^{2n+1})/(1-q)$	$1/(q)_n$
E(1)	1	$2(-1)^n q^{n^2}$	$1/((q)_n(-q)_n)$
E(3)	q	$(-1)^n q^{n^2}(1 - q^{2n+1})/(1-q)$	$1/((q)_n(-q)_n)$
E(4)	1	$(-1)^n q^{n^2}(q^n + q^{-n})$	$q^n/((q)_n(-q)_n)$

By applying Corollary 2.4 to these Bailey pairs, we have

$$
\begin{aligned}
\text{B(1)} \quad & \chi_{1,k+1}^{(2,2k+3)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k}(q)_{n_k}}. \\
\text{B(2)} \quad & \chi_{1,k}^{(2,2k+3)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + n_k}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k}(q)_{n_k}}. \\
\text{B(3)} \quad & \chi_{1,1}^{(2,2k+3)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + n_1 + \cdots + n_k}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k}(q)_{n_k}}. \\
\text{E(1)} \quad & \chi_{1,k+1}^{(2,2k+2)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k}(q)_{n_k}(-q)_{n_k}}. \\
\text{E(3)} \quad & \chi_{1,1}^{(2,2k+2)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + n_1 + \cdots + n_k}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k}(q)_{n_k}(-q)_{n_k}}. \\
\text{E(4)} \quad & \chi_{1,k}^{(2,2k+2)}(q) = \chi_{1,k/2}^{(4,k+1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + n_k}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k}(q)_{n_k}(-q)_{n_k}}.
\end{aligned}
\tag{A.1}
$$

Furthermore, by applying Corollary 2.6 to B(3), we obtain the interpolating expression between B(1) and B(3):

$$
\chi_{1,i}^{(2,2k+3)}(q) = \chi_{1,2k+3-i}^{(2,2k+3)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + n_i + \cdots + n_k}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k}(q)_{n_k}},
\tag{A.2}
$$

where $1 \leq i \leq k+1$.

From each form of (α, β), we can easily guess that E(1), E(4) and E(3) are the counterpart of B(1), B(2)

and B(3). Actually, using Corollary 2.6 we get a interpolating expression between E(1) and E(3)

$$\chi_{1,i+1}^{(2,2k+2)}(q) = \chi_{2,(i+1)/2}^{(4,k+1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2+\cdots+n_k^2+n_{i+1}+\cdots+n_k}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{n_k}(-q)_{n_k}}. \tag{A.3}$$

where $0 \leq i \leq k$.

Let us go on A series in her table [20]. In each case $\alpha_0 = 1$.

	a	α_{3r}	α_{3r-1}	α_{3r+1}	β_n
A(1)	1	$q^{6r^2-r}+q^{6r^2+r}$	$-q^{6r^2-5r+1}$	$-q^{6r^2+5r+1}$	$1/(q)_{2n}$
A(2)	q	q^{6r^2+r}	q^{6r^2-r}	$-q^{6r^2+5r+1}-q^{6r^2+7r+2}$	$1/(q^2)_{2n}$
A(3)	1	$q^{6r^2-2r}+q^{6r^2+2r}$	$-q^{6r^2-2r}$	$-q^{6r^2+2r}$	$q^n/(q)_{2n}$
A(4)	q	q^{6r^2+4r}	q^{6r^2-4r}	$-q^{6r^2+4r}-q^{6r^2+8r+2}$	$q^n/(q^2)_{2n}$
A(5)	1	$q^{3r^2-r}+q^{3r^2+r}$	$-q^{3r^2-r}$	$-q^{3r^2+r}$	$q^{n^2}/(q)_{2n}$
A(6)	q	q^{3r^2-r}	q^{3r^2+r}	$-q^{3r^2+r}-q^{3r^2+5r+2}$	$q^{n^2}/(q^2)_{2n}$
A(7)	1	$q^{3r^2-2r}+q^{3r^2+2r}$	$-q^{3r^2-4r+1}$	$-q^{6r^2+4r+1}$	$q^{n^2-n}/(q)_{2n}$
A(8)	q	q^{3r^2+2r}	q^{3r^2-2r}	$-q^{2r^2+2r}-q^{3r^2+4r+1}$	$q^{n^2+n}/(q^2)_{2n}$

Andrews [11] pointed out that Slater's A series consist of four Bailey pairs and their dual pairs. In fact, A(1) and A(5) are dual, the same holds between A(2) and A(8); A(3) and A(7); A(4) and A(6).

By applying Corollary 2.4 to A(1)–A(8) we obtain

$$
\begin{aligned}
\text{A(1)} \quad & \chi_{1,k+1}^{(3,3k+2)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2+\cdots+n_k^2}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k}}. \\[4pt]
\text{A(2)} \quad & \chi_{1,2k+1}^{(3,3k+2)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2+\cdots+n_k^2+n_1+\cdots+n_k}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+1}}. \\[4pt]
\text{A(3)} \quad & \chi_{1,k}^{(3,3k+2)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2+\cdots+n_k^2+n_k}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k}}. \\[4pt]
\text{A(4)} \quad & \chi_{1,2k+2}^{(3,3k+2)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2+\cdots+n_k^2+n_1+\cdots+n_{k-1}+2n_k}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+1}}. \\[4pt]
\text{A(5)} \quad & \chi_{1,k}^{(3,3k+1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2+\cdots+n_{k-1}^2+2n_k^2}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k}}. \\[4pt]
\text{A(6)} \quad & \chi_{1,2k}^{(3,3k+1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2+\cdots+n_{k-1}^2+2n_k^2+n_1+\cdots+n_k}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+1}}. \\[4pt]
\text{A(7)} \quad & \chi_{1,k+1}^{(3,3k+1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2+\cdots+n_{k-1}^2+2n_k^2-n_k}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k}}. \\[4pt]
\text{A(8)} \quad & \chi_{1,2k+1}^{(3,3k+1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2+\cdots+n_{k-1}^2+2n_k^2+n_1+\cdots+n_{k-1}+2n_k}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+1}}.
\end{aligned}
\tag{A.4}
$$

Furthermore, by applying Corollary 2.6 to A(2), we obtain the following expression

$$\sum_{j=0}^{i} q^j \chi_{1,2k-i+2j+1}^{(3,3k+2)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2+\cdots+n_k^2+n_{i+1}+\cdots+n_k}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{2n_k+1}}, \tag{A.5}$$

where $0 \leq i \leq k$. We can reproduce these equation (A.5) by setting $\kappa = 1$ in (3.9).

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